Spatial Durbin Mixture Models

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Outline

1 Motivations and Contributions
2 Spatial Durbin Mixture Model Specifications
3 Interpretation of the SDM-M
4 An Empirical Application
5 Conclusions and Additional Research
6 Appendix
Homogeneous models are often used to evaluate heterogeneous behavioral responses.
Motivations and Contributions

Motivation

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  - Nest both the standard SDM (G=1) and the HSDM (G=N) specification as special cases.
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- Conditional group assignment allows for interpretation of unobserved intra- and inter-group dynamics.
Mean varies across **clusters**.

- A form of fixed effects can be implemented by allowing intercept to vary.
- Approach requires delineation of data into smaller clusters which may or may not be feasible.

Figure: Spatial Heterogeneity
Spatial Heterogeneity

- Mean may or may not vary across groups \((g = 1, 2, \ldots, G)\)
- Fixed effects may be spurious.
- SAR model overstates spatial dependency.
- Delineation into small clusters is nearly impossible with any sort of accuracy.

Figure: Spatial Dependence/Mixture
Spatial Durbin Models

Spatial Durbin Model

\[ Y = \rho W Y + X \beta + WX \Phi + \epsilon \]  \hspace{1cm} (1)

\( W \) is an \( N \times N \) right-stochastic matrix which represents the connectivity structure for the sample. \( \rho \) is typically positive in homogeneous models but can be negative. Most other spatial specifications are nested within the structure of the SDM.
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Finite Mixture Models

G-component Mixture Model

- A mixture of Gaussian distributions can approximate most other distribution forms.
Finite Mixture Models

G-component Mixture Model

- A mixture of Gaussian distributions can approximate most other distribution forms.
- Generally can be written as:

\[
p(y_i | x_i, \beta, \Sigma, z, \pi) = \sum_{g=1}^{G} \pi_g N(y_i | \beta_g, \sigma^2_g), \quad \sum_{g=1}^{G} \pi_g = 1
\]  \hspace{1cm} (2)

\[
y_i = \sum_{g=1}^{G} z_{ig} \alpha_g + \sum_{k=1}^{K} \sum_{g=1}^{G} z_{ig} x_i^k \beta_g^k + \sum_{g=1}^{G} \Omega_{gg} z_{ig} \epsilon_i, \quad i = 1, 2, \ldots, N
\]  \hspace{1cm} (3)
Spatial Durbin Mixture Model Specifications

Spatial Durbin Mixture Model: SDM-M

\[ y_i = \sum_{g=1}^{G} z_i g \alpha_g + \sum_{g=1}^{G} z_i g \rho_g \sum_{j=1}^{N} W_{ij} y_j + \sum_{k=1}^{K} \sum_{g=1}^{G} z_i g x_i^k \beta_g^k + \sum_{k=1}^{K} \sum_{g=1}^{G} \sum_{j=1}^{N} W_{ij} z_i g x_j^k \phi_g^k + \sum_{g=1}^{G} \Omega_{gg} z_i g \epsilon_i \]  

(4)

Matrix Notation:

\[ y = \tilde{\alpha} + \tilde{\Psi} W y + \tilde{X} B + W \tilde{X} \Phi + \tilde{\epsilon} \] 

(5)

Definitions:

\[ \tilde{\alpha} = z \alpha \]

\[ \tilde{X} = (\iota'_G \otimes X) \odot (z \otimes \iota'_K) \]

\[ \tilde{\Psi} = z \psi, \psi = (\rho_1, \ldots, \rho_g) \]

\[ \tilde{\epsilon} = (z \Omega^{1/2}) \odot \epsilon \]
Spatial Durbin Error Mixture Model: SDEM-M

\[ y_i = \sum_{g=1}^{G} z_{ig} \alpha_g + \sum_{k=1}^{K} \sum_{g=1}^{G} z_{ig} x_{ik} \beta_{kg} + \sum_{k=1}^{K} \sum_{g=1}^{G} \sum_{j=1}^{N} W_{ij} z_{ig} x_{jk} \phi_{kg} + \eta_i \]  \hspace{1cm} (6)

\[ \eta_i = \sum_{g=1}^{G} z_{ig} \lambda_g \sum_{j=1}^{N} W_{ij} \eta_j + \sum_{g=1}^{G} z_{ig} \epsilon_i \]

Matrix notation:

\[ y = \tilde{\alpha} + \tilde{X}B + W\tilde{X}\Phi + (I - \tilde{\Psi}W)^{-1}\tilde{\epsilon} \]  \hspace{1cm} (7)
The Parameters of Interest

- $B, \Phi$ - Each are $KG \times 1$ vectors of coefficients.
- $\Omega$ - $G \times 1$ vector of variances.
- $\psi$ - $G \times 1$ vector of scalars indicating strength of spatial dependence.
- $\omega, z$ - An $N \times G$ matrix indicating which group each region is in.
- $\pi$ - A $G \times 1$ vector of group weights over the sample.
Sampling Algorithm

- Set initial values for parameters.
- Expand $X$ to $\tilde{X}$.
- Draw from $p(\tilde{B} | \Omega, \psi, z, \pi, x, y) \sim N(D_{\tilde{B}}d_{\tilde{B}}, D_{\tilde{B}})$
- Draw from $p(\Omega | \tilde{B}, \psi, z, \pi, z, y) \sim IG(c, C)$
- Draw from (M-H Step) $p(\rho_g | \rho_{-g}, \Omega, \tilde{B}, z, \pi, x, y) \propto |I_N - \tilde{\Psi}| \exp\left[-\frac{1}{2}e'\Omega^{-1}e\right]$
- Draw from $p(z_i | \Omega, \tilde{B}, \psi, x, y) \sim MN(1, [\omega_{i1}, \ldots, \omega_{iG}])$
- Draw from $p(\pi | \Omega, \tilde{B}, \psi, z, x, y) \sim D(\alpha + N)$
- Iterate
Partial Derivatives for the SDM-M

For reference the partial derivatives of the SDM are:

$$\frac{\delta y}{\delta x^k} = (I - \rho W)^{-1}(I \beta^k + W \phi^k)$$

(8)

- The partial derivative is, by definition, an $N \times N$ matrix.
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- This matrix is summarized by total, direct, and indirect effects. (Lesage & Pace, 2009)
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**Total Effects:**
- Total impact to an observation given by averaging column sum vector $c_k = S_k(W)\iota_n$, or $n^{-1}\iota_n'c_k$. 
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**Direct Effects:** \( n^{-1}tr(S_k(W)) \)
Interpretation of the SDM-M

Partial Derivatives for the SDM-M

For reference the partial derivatives of the SDM are:

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- **Total Effects:**
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  - Total impact from an observation given by averaging row sum vector $r_k = \iota_n' S_k(W)$, or $n^{-1} r_k \iota_n$.
- **Direct Effects:** $n^{-1} tr(S_k(W))$
- **Indirect Effects:** Total - Direct

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SRSA Conference 2017  
March 27, 2017
The "To" and "From" Now Matters

The partial derivative for the SDM-M:

$$\frac{\delta y}{\delta x^k} = (I - \tilde{\Psi} W)^{-1}\left(\text{diag}(z_{\beta_g}^k) + \text{diag}(z_{\phi_g}^k) W\right)$$

(9)

- The partial derivative is still an $N \times N$ matrix.
- Let $M_k(W) = (I - \tilde{\Psi} W)^{-1}\left(\text{diag}(z_{\beta_g}^k) + \text{diag}(z_{\phi_g}^k) W\right)$.
- Now $n^{-1}l'_n c_r \neq n^{-1}r_r l_n$. 
Relative Location
Defining Interior and Border

Interior (Λ)

\[ i \in \lambda_g \iff j \in g \ \forall \ w_{ij} > 0 \] (10)

Border (Γ)

\[ i \in \gamma_g \ \exists \ j \notin g \ \forall \ w_{ij} > 0 \] (11)

\[ \sum_{g=1}^G \lambda_g = \Lambda, \sum_{g=1}^G \gamma_g = \Gamma, \ \Lambda \cup \Gamma = N \]
Depth of Neighbors

- Since we are interested in differentiating effects for interior and border regions the "depth" of the weight matrix matters.
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The more regions included as first order neighbors the more isolated a region must be to be considered interior.
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The more regions included as first order neighbors the more isolated a region must be to be considered interior.

This means that some groups may consist entirely of border agents while others are a mix.
Interpretation of the SDM-M

Interior and Border Effects Realized

As a reminder $M_r(W) = (I - \tilde{\Psi} W)^{-1} \left( \text{diag}(z_{\beta g}^k) + \text{diag}(z_{\phi g}^k) W \right)$.

**Group Interior Effects:**

\[ DE^\lambda_g = (n_g^\lambda)^{-1} \sum_{i=1}^{n_g^\lambda} m_{ii} \]  \hspace{1cm} (12)

\[ SI^\lambda_g = (n_g^\lambda)^{-1} \sum_{i=1}^{n_g^\lambda} \sum_{j=1, j \neq i}^{n_g^\lambda} m_{ij} \]  \hspace{1cm} (13)

\[ SO^\lambda_g = (n_g^\lambda)^{-1} \sum_{i=1}^{n_g^\lambda} \sum_{i=1, i \neq j}^{n_g^\lambda} m_{ij} \]  \hspace{1cm} (14)

**Group Border Effects:**

\[ DE^\gamma_g = (n_g^\gamma)^{-1} \sum_{i=1}^{n_g^\gamma} m_{ii} \]  \hspace{1cm} (15)

\[ SI^\gamma_g = (n_g^\gamma)^{-1} \sum_{i=1}^{n_g^\gamma} \sum_{j=1, j \neq i}^{n_g^\gamma} m_{ij} \]  \hspace{1cm} (16)

\[ SO^\gamma_g = (n_g^\gamma)^{-1} \sum_{i=1}^{n_g^\gamma} \sum_{i=1, i \neq j}^{n_g^\gamma} m_{ij} \]  \hspace{1cm} (17)
Interpretation of the SDM-M

A Simple Example

- 8 Regions
- 2 Groups
- Contiguity Weight Matrix - Queen

**Group 1:**
- A, B, C, D
- $\rho_1 = -0.2$
- $\beta_1 = 1$, $\phi_1 = 0$

**Group 2:**
- E, F, G, H
- $\rho_2 = 0.5$
- $\beta_2 = 1$, $\phi_2 = 0$
The SDM-M reports group level border and interior effects.

- Provides more information than SDM.
- What happens if we vary $\rho_2$ over the domain while holding $\rho_1$ constant?
Interpretation of the SDM-M

Why interior and border? A visual justification...

Figure: Group 1 Responses

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Heterogeneity in Income

- Drawing from existing literature (Gu & Koenker, 2015).
An Empirical Application

Heterogeneity in Income

- Drawing from existing literature (Gu & Koenker, 2015).
- Utilizing the Panel Study of Income Dynamics (PSID).

Model:
\[
\log(\text{income}) = \tilde{\Psi} W \log(\text{income}) + X \beta + \Phi W X + \epsilon
\]

- \(X\) includes age, education, gender, race, marital status and home ownership.
- Block-diagonal (State) nearest-neighbor (5) weight matrix.
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- Drawing from existing literature (Gu & Koenker, 2015).
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  - N = 12,443

Model:
\[
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- Model: $\log(\text{income}) = \tilde{\Psi} W \log(\text{income}) + XB + WX\Phi + \epsilon$
  - $X$ includes age, education, gender, race, marital status and home ownership.
  - Block-diagonal (State) nearest-neighbor (5) weight matrix.
Heterogeneity in Income

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**Model:**

$$\log(\text{income}) = \Psi W \log(\text{income}) + XB + WX\Phi + \epsilon$$

- $X$ includes age, education, gender, race, marital status and home ownership.
- Block-diagonal (State) nearest-neighbor (5) weight matrix.
Results Summary

- Three distinct groups emerge from the data.
  - Group 1 - Fully-employed
  - Group 2 - Un-employed
  - Group 3 - Under-employed

- Spill-out effects from education tend to be positive even if the direct effects are not.
- Spill-in effects from education tend to be negative for both those that are un-employed and under-employed.
- Household dynamics become apparent by a *post hoc* analysis of the data under the estimated groupings.
- Estimates are robust to changes in the weight matrix.
  - Nearest Neighbor - 2 through 6.
  - Contiguity - Queen.
An Empirical Application

Parameter Estimates

- All hypothesis are examined under 95% HPD.
- Many of the spatially lagged characteristics have significant estimates ($\Phi$).
- Three Groups
  - Group One (*Fully-Employed*) - 80.4%
  - Group Two (*Unemployed*) - 14.3%
  - Group Three (*Under-Employed*) - 5.3%
Posteriors for $\Psi$

- Odds against (Mills, WP)
  $\rho_1 = 0 \approx 346,000 : 1$
- Odds against (Mills, WP)
  $\rho_2 = 0 \approx 1.19 : 1$
- Odds against (Mills, WP)
  $\rho_3 = 0 \approx 21.5 : 1$
Select Results: Education

- Obtaining additional human capital has large and positive impacts on income (Psacharopoulos, 1994; Montenegro & Patrinos, 2013; etc.)

- Agents in group 2 experience a negative spill-in when their neighbors acquire additional human capital.

- Agents in group 3 experience downward pressure on wages as a result of obtaining additional human capital.

<table>
<thead>
<tr>
<th>Group</th>
<th>Interior Direct</th>
<th>Estimate</th>
<th>Std. Dev.</th>
<th>Lower 95</th>
<th>Upper 95</th>
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<td>0.0101</td>
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</tr>
</tbody>
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A Quick Summary

- The spatial mixture class of models fills a gap between traditional spatial models (see LeSage & Pace 2009) and more recent developments in heterogeneous models (see Aquaro, et al., 2015 and LeSage & Chih, 2016).
A Quick Summary

- The spatial mixture class of models fills a gap between traditional spatial models (see LeSage & Pace 2009) and more recent developments in heterogeneous models (see Aquaro, et al., 2015 and LeSage & Chih, 2016).
- The SDM-M and SDEM-M nest many other functional forms as special cases.
The spatial mixture class of models fills a gap between traditional spatial models (see LeSage & Pace 2009) and more recent developments in heterogeneous models (see Aquaro, et al., 2015 and LeSage & Chih, 2016).

The SDM-M and SDEM-M nest many other functional forms as special cases.

The focus on group and locational dynamics gives avenues for model exploration that previously were not available.
Thank you!
### Appendix

#### Parameter Estimates

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### Partial Effects Summaries: Continued

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Conditional Distributions: SDM-M

1. Set initial values for parameters.

2. Expand $X$ to $\tilde{X}$.
   - $\tilde{X} = (\iota_G' \otimes X) \otimes (z \otimes \iota_K')$

3. $p(\tilde{B} | \Omega, \psi, z, \pi, x, y) \sim N(D_{\tilde{B}}d_{\tilde{B}}, D_{\tilde{B}})$
   - $D_{\tilde{B}} = (\tilde{X}' \Omega^{-1} \tilde{X} + V_{\tilde{B}})$
   - $d_{\tilde{B}} = \tilde{X}' \Omega^{-1} \tilde{y} + V_{\tilde{B}} \bar{B}_0$
   - $\tilde{y} = (I_N - \tilde{\Psi} W)y$

4. $p(\Omega | \bar{B}, \psi, z, \pi, z, y) \sim IG(c, C)$
   - $C = a + \frac{N}{2}$
   - $c = b + \frac{1}{2} e'e$
   - $e = \tilde{y} - \bar{X}\bar{B}$

- $p(\rho_g | \rho_{-g}, \Omega, \tilde{B}, z, \pi, x, y) \propto |I_N - \tilde{\Psi}| \exp \left[ \frac{-1}{2} e' \Omega^{-1} e \right]$
- $p(z_i | \Omega, \tilde{B}, \psi, x, y) \sim MN(1, [\omega_{i1}, \omega_{i2}, \ldots, \omega_{iG}])$
  - $\omega_{ig} = \frac{q_{ig}}{\sum_{g=1}^{G} q_{ig}}$
  - $q_{ig} = (2\pi\sigma_g^2)^{-1/2} \exp \left[ -\frac{1}{2\sigma_g^2} \left( y_i - \rho_g \sum_{j=1}^{N} w_{ij} y_j - x_i \beta_g - \phi_g \sum_{j=1}^{N} w_{ij} x_i \right)^2 \right]$
- $p(\pi | \Omega, \bar{B}, \psi, x, y, z) \sim D(\alpha + N)$